Toy multiverses of set theory

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Early evidence of relativity

Whatever your philosophical position on set theory, the following is an indisputable fact.

> *From the middle of 20th century onward, set theory has been the study of a multiverse of mathematical worlds.*

Developments in the early 20th century already hinted at the relativity to come.

Löwenheim-Skolem Theorem: (1920) If ZFC is consistent, then there is a countable model of ZFC.

Countability is not absolute: an uncountable set can become countable in a *better* model.

Compactness Theorem (Gödel, 1930): If ZFC is consistent, then there is a model of ZFC with ill-founded natural numbers.

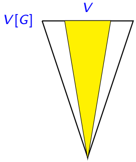
Well-foundedness is not absolute: a well-founded relation can become ill-founded in a *better* model.

Gödel's First Incompleteness Theorem: (1931) No computable extension of ZFC can decide the truth of all set-theoretic assertions.

Formal mathematics cannot produce an absolute notion of set.

Forcing

Cohen (1963) introduced the technique of forcing for enlarging a universe $V \models \mathrm{ZFC}$ to another universe, the forcing extension $V[G] \models \mathrm{ZFC}$, satisfying some desired properties.



- V is transitive in $V[G]$: no new sets are added to old sets.
- $V[G]$ wider but not taller than V: $\mathrm{Ord}^V = \mathrm{Ord}^{V[G]}$.

In a forcing extension:

- Continuum can be any cardinal of uncountable cofinality.
- Continuum function on the regular cardinals can assume any desired definable pattern (modulo necessary constraints).
- A Suslin line can exist or not.
- There can be (Δ^1_3) non-constructible reals.
- Cardinal characteristics of the continuum can have various values and relationships.

These and many other fundamental properties of sets can be changed by passing to a larger universe, a forcing extension.

Large cardinals

A large cardinal axiom asserts that there exists a cardinal with some "largeness" properties that make its existence unprovable in ZFC.

The large cardinal axioms form a hierarchy by consistency strength (often as well by implication) against which the consistency strength of any other set theoretic assertion can be measured.

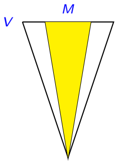
Set theorists continue to introduce new large cardinal notions.

Large cardinal axioms imply existence of transitive sub-universes (set-sized and class-sized).

- κ is inaccessible: $V_\kappa \models \mathrm{ZFC}$.
- κ is measurable: V has a proper class transitive sub-universe M into which it elementarily embeds.
- κ is supercompact: for every cardinal λ, V has a proper class transitive sub-universe M, closed under λ-sequences, into which it elementarily embeds.

Inner models

An inner model M of a universe $V \models \mathrm{ZFC}$ is a transitive class sub-universe containing all the ordinals.



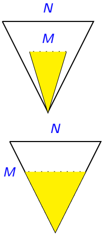
A typical universe will have many inner models.

- Canonical inner models: Gödel's L, the Dodd-Jensen core model K, Woodin's minimal models M_n for n-Woodin cardinals.
  - Construction follows a recipe according to which sets are built bottom-up.
  - Bounds on large cardinals.
  - Some forcing absoluteness.
- HOD: hereditarily ordinal definable sets.
- Targets of elementary embeddings: well-founded ultrapowers of the universe.
- Grounds: W such that there is a forcing notion $\mathbb{P} \in W$ and $G \in V$ with $W[G] = V$.
- Ord-length direct limits of countable iterable models.

Model theoretic constructions

Definition: Suppose $M \subseteq N$ are models of ZFC.

- N is an end-extension of M, denoted $M \subseteq_e N$ if M is transitive in N.
- N is a top-extension of M, denoted $M \subseteq_t N$ if every element of $N \setminus M$ has higher rank than all elements of M.
  - The ordinals of M may not have a least upper bound in N.
  - M is covered by some V_α^N.



Keisler-Morley Extension Theorem: Every countable model $M \models \mathrm{ZFC}$ has a proper elementary top-extension N.

Barwise Extension Theorem: Every countable model $M \models \mathrm{ZFC}$ has an end-extension to a model $N \models \mathrm{ZFC} + V = L$.

Non-well-founded ultrapowers of the universe

- For every ultrafilter U, V can define the ultrapower V/U into which it elementarily embeds (usually with ill-founded natural numbers).

The Universist Position

There is only one true universe of set theory, the mathematical world where all mathematics takes place.

- The incompleteness phenomenon is a by-product of formal methods.
- All the universes built by set theorists are part of a quest to understand the properties of the one true universe.
- A deeper understanding of the structure of sets should yield an intuition about the true universe.
- (Woodin) The true universe will be a canonical model with a recipe for construction justifying why each set ends up in it.
- The true universe will have all known large cardinals.
- The true universe will satisfy a computable theory such that statements independent of that theory will be esoteric assertions irelevent to the fundamental structure of sets.

Goal: Use what set theorists learned from the universes they have built to understand the properties of the true universe of set theory.

The Universist Position in Arithmetic

Peano Arithmetic PA (1889): axiomatization of natural numbers consisting of basic properties of addition and multiplication together with the induction scheme.

There is only one true model of PA, the standard model $\mathbb{N}$.

- We have enough intuition about the natural numbers to be certain of this conclusion.
- We study non-standard models of PA to understand the Peano Axioms as a formal theory.
- Statements naturally encountered in number theory are decided by PA.
- Independent statements are difficult to find and artificial in nature.

The Multiversist Position

There is no absolute set-theoretic background. There is a multiverse of universes of set theory, mathematical worlds that have equal ontological status.

- Forcing extensions, canonical and non-canonical inner models, universes with and without large cardinals.
- Each of these universes instantiates a different concept of set and all these concepts of set are equally valid.

Goal: Study what kinds of universes can exist and how they are related to one another.

Questions:

- Should all models in the multiverse have the same height?
- Do ill-founded models belong in the multiverse?
- Do models of fragments of ZFC (e.g. ZF) belong in the multiverse?

The Radical Multiversist

(Hamkins) Without an absolute set theoretic background, there cannot be an absolute notion of countability or well-foundedness. The relativity of the notion of set must extend to the notions of well-foundedness, height, and even the natural numbers.

- Every universe will be revealed to be countable and ill-founded from the perspective of a *better* universe.
- The natural numbers of a given universe will be revealed to be ill-founded from the perspective of a *better* universe.
- The Universist Position on the natural numbers is false.

The Radical Multiversist position is captured by Hamkins' Multiverse Axioms.

The Hamkins Multiverse Axioms

Realizability: If M is a universe and N is a definable class in M such that M believes that $N \models \mathrm{ZFC}$, then N is a universe.

- If $m \in M$ and M believes that $m \models \mathrm{ZFC}$, then m is in the multiverse.
- If κ is inaccessible in M, then V_κ^M is in the multiverse.
- The constructible universe L^M is in the multiverse.
- If $j : M \to M^*$ is an elementary embedding in M, then (possibly ill-founded) M^* is in the multiverse.

Forcing Extension: If M is a universe and $\mathbb{P} \in M$ is a forcing notion, then there is a universe $M[G]$ where $G \subseteq \mathbb{P}$ is M-generic.

Class Forcing Extension: If M is a universe and $\mathbb{P}$ is a definable ZFC-preserving forcing notion in M, then there is a universe $M[G]$ where $G \subseteq \mathbb{P}$ is V-generic.

Reflection: If M is a universe, then there is a universe N such that $M \prec V_\theta^N \prec N$ for some rank initial segment V_θ^N of N.

- Recall the Keisler-Morley Extension Theorem.
- Reflection as a guiding principle in set theory.

The Hamkins Multiverse Axioms (continued)

Countability: Every universe M is a countable set in another universe N.

Well-founded Mirage: Every universe M is a set in another universe N which thinks that $\mathbb{N}^M$ is ill-founded.

Reverse Embedding: If M is a universe and $j : M \to N$ is an elementary embedding definable in M, then there is a universe M^* and an elementary embedding $j^* : M^* \to M$ definable in M^* such that $j = j^*(j^*)$.

- Every elementary embedding has already been iterated many times.

Absorbtion into L: Every universe M is a transitive set in another universe N satisfying $V = L$.

- Recall the Barwise Extension Theorem.

Toy multiverses

Without attempting to find a formal background for studying a multiverse of models of set theory, we can study toy multiverses inside models of ZFC.

Question: Supposing ZFC to be consistent, is there a collection of models of ZFC satisfying the Hamkins Multiverse Axioms?

Question:

- Is there a natural such collection?
- Does the the collection of all countable models $M \models \mathrm{ZFC}$ with ill-founded $\mathbb{N}^M$ satisfy the Hamkins Multiverse Axioms?

Theorem: (G., Hamkins) Supposing ZFC to be consistent, the collection of all countable computably saturated models of ZFC satisfies the Hamkins Multiverse Axioms.

Computably saturated models

Let $\mathcal{L}$ be a computable first-order language.

Definition: A type $p(\bar{x},\bar{y})$ in $\mathcal{L}$ is computable if the collection of Gödel codes of formulas in the type

$$\{\ulcorner\varphi\urcorner \mid \varphi(\bar{x},\bar{y}) \in p(\bar{x},\bar{y})\}$$

is a computable set.

Definition: A model M of $\mathcal{L}$ is computably saturated if whenever $p(\bar{x},\bar{y})$ is a computable type and a tuple $\bar{a} \in M$ such that $p(\bar{x},\bar{a})$ is finitely realizable in M, then $p(x,\bar{a})$ is realized in M: there is a tuple $\bar{b} \in M$ such that for all $\varphi(\bar{x},\bar{a}) \in p(\bar{x},\bar{a})$, $M \models \varphi(\bar{b},\bar{a})$.

Proposition: Every model M of $\mathcal{L}$ of cardinality κ has a computably saturated elementary extension $M \prec N$ of cardinality κ.

Proof: Close under elements realizing computable types in ω-many steps. □

Proposition: If $M \models \mathrm{ZFC}$ is computably saturated, then $\mathbb{N}^M$ is ill-founded.

Proof: Let $p(x) = \{x > n \mid n \in \mathbb{N}\}$ assert that x is greater than every standard natural number. □

Standard system

Definition: Suppose $M \models \mathrm{ZFC}$ has an ill-founded $\mathbb{N}^M$. The standard system

$$\mathrm{SSy}(M) = \{A \cap \mathbb{N} \mid A \in M\}$$

consists of the traces of sets in M on the (true) natural numbers $\mathbb{N}$.

- For every nonstandard $b \in \mathbb{N}^M$, $\mathrm{SSy}(M) = \{A \cap \mathbb{N} \mid A \in M,\ A \subseteq^M b\}$.
- $\mathrm{SSy}(M)$ is a Boolean algebra of subsets of $\mathbb{N}$.
- $\mathrm{SSy}(M)$ is closed under computability: If $A \in \mathrm{SSy}(M)$ and B is computable from A, then $B \in \mathrm{SSy}(M)$.
- If T is an infinite binary tree (coded) in $\mathrm{SSy}(M)$, then $\mathrm{SSy}(M)$ has a branch through T.
  - If $\mathcal{T}$ is a consistent first-order theory, then there is $\mathcal{T}$-computable tree T such that every branch through T is a consistent completion of $\mathcal{T}$.
  - If $\mathcal{T}$ is (coded) in $\mathrm{SSy}(M)$, then $\mathrm{SSy}(M)$ has a consistent completion of $\mathcal{T}$.

Proposition: If M and N are models of ZFC with ill-founded natural numbers such that $\mathbb{N}^N$ is an end-extension of $\mathbb{N}^M$, $\mathbb{N}^M \subseteq_e \mathbb{N}^N$, then $\mathrm{SSy}(M) = \mathrm{SSy}(N)$.

Types in the standard system

Proposition: If $M \models \mathrm{ZFC}$ is computably saturated, then for every $\bar{a} \in M$,

$$\mathrm{tp}(\bar{a}) = \{\varphi(\bar{x}) \mid M \models \varphi(\bar{a})\}$$

is in $\mathrm{SSy}(M)$.

Proof: Let $p(x, \bar{y})$ be the computable type consisting of assertions $\varphi(\bar{y}) \leftrightarrow \ulcorner\varphi\urcorner \in x$ for every formula $\varphi(x, \bar{y})$.

- x codes all formulas true of $\bar{y}$.
- $p(x, \bar{a}) = \{\varphi(\bar{a}) \leftrightarrow \ulcorner\varphi\urcorner \in x \mid \varphi(x, \bar{y}) \text{ is a formula}\}$ is finitely realizable because any finite set is coded.
- Let b realize $p(x, \bar{a})$ in M. Then $b \cap \mathbb{N} = \mathrm{tp}(\bar{a})$ is in $\mathrm{SSy}(M)$. $\square$

In particular, the theory of M, $\mathrm{Th}(M)$, is in $\mathrm{SSy}(M)$.

Standard system saturation

Definition: (Wilmers) Suppose $M \models \mathrm{ZFC}$ has ill-founded $\mathbb{N}^M$. Then M is $\mathrm{SSy}(M)$-saturated if whenever $p(\bar{x}, \bar{y})$ is in $\mathrm{SSy}(M)$ and $\bar{a} \in M$ such that $p(\bar{x}, \bar{a})$ is finitely realizable, then $p(\bar{x}, \bar{a})$ is realized in M.

If M is $\mathrm{SSy}(M)$-saturated, then M is computably saturated because $\mathrm{SSy}(M)$ has all the computable sets.

Proposition: (Wilmers) If $M \models \mathrm{ZFC}$ is computably saturated, then M is $\mathrm{SSy}(M)$-saturated.

Proof: Suppose $p(\bar{x}, \bar{y})$ is in $\mathrm{SSy}(M)$: $A \in M$ such that $\varphi(\bar{x}, \bar{y}) \in p(\bar{x}, \bar{y})$ iff $\ulcorner \varphi \urcorner \in A$.

- Let $p^*(\bar{x}, \bar{y}, z)$ be the computable type consisting of assertions $\varphi(\bar{x}, \bar{y}) \leftrightarrow \ulcorner \varphi \urcorner \in z$ for every formula $\varphi(\bar{x}, \bar{y})$.
- If $p(\bar{x}, \bar{a})$ is finitely realizable, then so is $p^*(\bar{x}, \bar{a}, A)$, so $p^*(\bar{x}, \bar{a}, A)$ is realized. $\square$

Characterizing countable computably saturated models

Theorem: (Folklore) Suppose M and N are countable computably saturated models of ZFC. Then $M \cong N$ if and only they have the same theory and the same standard system.

Proof: Back and forth argument using:

- $\mathrm{SSy}(N)$ codes types of all elements of M and visa-versa.
- Both models are standard system saturated. $\square$

Models living inside ill-founded models

Theorem: If $N \models \mathrm{ZFC}$ has ill-founded $\mathbb{N}^N$ and $M \models \mathrm{ZFC}$ is an element of N, then M is computably saturated.

Proof: Let $p(\bar{x}, \bar{a})$ be a finitely realizable computable type over M.

- There is $A \in N$ such that $\varphi(\bar{x}, \bar{y}) \in p(\bar{x}, \bar{y})$ iff $\ulcorner\varphi\urcorner \in A$ ($\mathrm{SSy}(N)$ has all computable sets).
- N has a truth predicate for M.
- For every standard natural number n, N knows that there is $\bar{b} \in N$ such that $M \models \varphi(\bar{b}, \bar{a})$ for every φ with $\ulcorner\varphi\urcorner \in A \cap n$.
- There is a nonstandard natural number $c \in N$ and $\bar{b} \in M$ such that N thinks that $M \models p(\bar{b}, \bar{a})$ for every φ with $\ulcorner\varphi\urcorner \in A \cap c$.
- N is correct that $M \models \varphi(\bar{b}, \bar{a})$ for every standard φ.
- $\bar{b} \in M$ realizes $p(\bar{x}, \bar{a})$. $\square$

Corollary: A toy multiverse satisfying the Hamkins Multiverse Axioms must consist of computably saturated models.

The multiverse of computably saturated models

Theorem: If $N \models \mathrm{ZFC}$ is countable and computably saturated, then there is $M \cong N$ such that $M \in N$ and N thinks that M is countable with ill-founded natural numbers.

Proof: Let $A \in N$ such that $\varphi \in \mathrm{Th}(N)$ iff $\ulcorner\varphi\urcorner \in A$ ($\mathrm{Th}(N) \in \mathrm{SSy}(N)$).

- By the Reflection Theorem, N thinks that (the theory coded by) $A \cap n$ is finitely realizable for every standard natural number n.
- N has a countable model of (the theory coded by) $A \cap c$ for some nonstandard c.
- N has a countable model M of $A \cap c$ with ill-founded natural numbers.
- $M \models \mathrm{Th}(N)$ and $\mathrm{SSy}(N) = \mathrm{SSy}(M)$ because $\mathbb{N}^M \subseteq_e \mathbb{N}^N$.
- $M \cong N$. $\square$

Corollary: The collection of all countable computably saturated models of ZFC satisfies:

- Countability
- Well-founded Mirage
- Realizability
- Forcing and Class Forcing Extension

The multiverse of computably saturated models (continued)

Reflection: If M is a universe, then there is a universe N such that $M \prec V_\theta^N \prec N$ for some rank initial segment V_θ^N of N.

Theorem: (Ressayre) If $M \models \mathrm{ZFC}$ is computably saturated, then some rank initial segment $V_\alpha^M \prec M$.

Proof: Let $p(x)$ consist of assertions

- $\forall \ulcorner\varphi\urcorner \in \Sigma_n \, \mathrm{Tr}_{\Sigma_n}(\ulcorner\varphi\urcorner) \leftrightarrow \mathrm{Tr}_{\Delta_0}(\ulcorner x \models \varphi\urcorner)$ for every n
- "x is a rank initial segment".

$p(x)$ is finitely realized in M by the Reflection Theorem. □

Corollary: The collection of all countable computably saturated models of ZFC satisfies Reflection.

Proof: Let $M \models \mathrm{ZFC}$ be a countable computability saturated model.

- $V_\alpha^M \prec M$ for some α.
- $V_\alpha \cong M$ since they have the same theory and standard system. □

The multiverse of computably saturated models (continued)

Absorbtion into L: Every universe M is a transitive set in another universe N satisfying $V = L$.

Theorem: The multiverse of countable computably saturated models of ZFC satisfies Absorbtion into L.

Proof: Let $M \models \mathrm{ZFC}$ be a countable computably saturated model.

- Let $A \in M$ be such that $\varphi \in \mathrm{Th}(M)$ iff $\ulcorner\varphi\urcorner \in A$.
- By the Reflection Theorem, M satisfies, for every standard natural number n, that there is a countable transitive model of $A \cap n$.
- By Shoenfield's Absoluteness Theorem $N = L^M$ satisfies, for every true natural number n, that there is a countable transitive model of $A \cap n$.
- $\mathrm{SSy}(N) = \mathrm{SSy}(M)$ since they have the same natural numbers.
- Let $B \in N$ be such that $A \cap \mathbb{N} = B \cap \mathbb{N}$.
- There is a nonstandard natural number $c \in N$ such that N has a transitive model K of $B \cap c$.
- $\mathrm{SSy}(K) = \mathrm{SSy}(N) = \mathrm{SSy}(M)$.
- $K \cong M$. $\square$

The multiverse of computably saturated models (continued)

Reverse Embedding: If M is a universe and $j : M \to N$ is an elementary embedding definable in M, then there is a universe M^* and an elementary embedding $j^* : M^* \to M$ definable in M^* such that $j = j^*(j^*)$.

Theorem: The collection of all countable computably saturated models of ZFC satisfies Reverse Embedding.

Proof: Let $M \models \mathrm{ZFC}$ be a countable computably saturated model and let $j : M \to N$ be a definable elementary embedding in M.

- Suppose $j : M \to N$ is defined by $\varphi(x, y, a)$.
- $j(j) : N \to K$ is defined in N by $\varphi(x, y, j(a))$.
- $N \cong M$ since they have the same theory and standard system.
- Since $\mathrm{tp}(a) = \mathrm{tp}(j(a))$, there is $\pi : M \xrightarrow{\cong} N$ with $\pi(a) = j(a)$.
- $(M, N, j) \cong (N, K, j(j))$.
- Take pre-image of $M \xrightarrow{j} N \xrightarrow{j(j)} K$ under π to obtain $M^* \xrightarrow{h} M \xrightarrow{h(h)} N$. $\square$

Weak Well-founded Mirage

Weak Well-founded Mirage: Every universe M is a set in another universe which thinks that its membership relation is ill-founded.

Supposing there is a transitive model of ZFC, the Cohen-Shepherdson model L_α is the minimum transitive model of ZFC.

Every model of ZFC in L_α is ill-founded.

Theorem: (G., Godziszewki, Meadows, Williams) The collection in L_α of all models M such that L_α thinks that M is a countable model of ZFC satisfies:

- Realizability
- Set Forcing and Class Forcing Extension
- Countability
- Weak Well-founded Mirage

Multiverse with weak well-founded mirage

Definition: Let A be a countable admissible set. The language $\mathcal{L}_A = L_{\omega_1,\omega} \cap A$.

Barwise Completeness Theorem: Let A be a countable admissible set and let $\mathcal{T}$ be a theory in $\mathcal{L}_A$ that is Σ_1-definable over A. Then TFAE:

- $\mathcal{T}$ is satisfiable.
- $\mathcal{T}$ is consistent.
- A thinks that $\mathcal{T}$ is consistent.

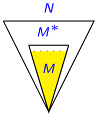
Proof of Theorem: Let $M \models \mathrm{ZFC}$ be countable in L_α.

- In L_α, let A be a countable admissible set with $M \in A$.
- Let $\mathcal{T}$ be theory ZFC together with the $L_{\omega_1\omega}$-atomic diagram of A.
- $\mathcal{T}$ is Σ_1-definable over A.
- $\mathcal{T}$ is satisfiable since $L_\alpha \models \mathcal{T}$.
- By Barwise Completeness Theorem, A thinks that $\mathcal{T}$ is consistent.
- By Barwise Completeness Theorem inside L_α, there is $N \models \mathcal{T}$.
- $M \in N$ and N end-extends A.
- M is ill-founded in N since A sees that M is ill-founded and N end-extends A. □

Covering multiverse axioms

Covering Countability: For every universe M, there is a universe N and a countable set $A \in N$ with $M \subseteq A$.

Covering Well-founded Mirage: For every universe M, there is a universe N and a model $M^* \in N$ end-extending M such that N thinks that $\mathbb{N}^{M^*}$ is ill-founded.



Covering Absorbtion into L: For every universe M, there is a universe $N \models \mathrm{ZFC} + V = L$ and a model $M^* \in N$ end-extending M.

Theorem: (G., Godziszewki, Meadows, Williams) There is a toy multiverse, not all of whose models are computably saturated, satisfying:

- Realizability
- Forcing Extension
- Class Forcing Extension for Ord-cc forcing
- Covering Countability
- Covering Well-founded Mirage
- Covering Absorbtion into L