The old and the new of virtual large cardinals

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A large cardinal chart

Kunen's Inconsistency

$j : V_{\lambda+2} \to V_{\lambda+2}$ with $\text{crit}(j) = \kappa$ — $0 = 1$

$j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$ $(\lambda \leq \bigcup_{n<\omega} j^n(\kappa) + 1)$ — rank-into-rank

for every $\lambda > \kappa$, $j : V_\lambda \to V_\eta$ with $\text{crit}(j) = \kappa$, and $j(\kappa) > \lambda$ — extendible

for every $\lambda > \kappa$, $j : V \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$ $(j \restriction \lambda \in M)$ — supercompact
for every $\lambda > \kappa$, there is $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$

for every $\lambda > \kappa$, $j : V \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$ — strong
for every $\lambda > \kappa$, $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$

$j : V \to M$ with $\text{crit}(j) = \kappa$ — measurable
for every $\lambda > \kappa$, $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$

L

ω-Erdős

completely ineffable

ineffable

weakly compact

inaccessible

Large cardinal embeddings in a forcing extension

Question: What happens if we ask that elementary embeddings characterizing a given large cardinal exist in a forcing extension of V?

Versions of Measurability

In a forcing extension $V[G]$:

- There is $j : V \to M$ with $\text{crit}(j) = \kappa$ and $M \subseteq V$.
  - This is not formalizable (no equivalent ultrafilter definition).
  - Equiconsistent with a measurable cardinal.
  - Is κ measurable in V?
- (generically measurable) There is $j : V \to M \subseteq V[G]$ with $\text{crit}(j) = \kappa$.
  - Has an equilavent ultrafilter definition.
  - Equiconsistent with a measurable cardinal.
  - κ can be a small cardinal like ω_1.
- (virtually measurable) For every $\lambda > \kappa$, there is $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$ and $M \subseteq V$.
  - Equivalently $M \in V$.
  - κ is completely ineffable and more.
  - Downward absolute to L.

Large cardinal embeddings in a forcing extension (continued)

Versions of Supercompactness

For every $\lambda > \kappa$, in a forcing extension $V[G]$:

- There is $j : V \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $j \restriction \lambda \in M$ and $M \subseteq V$ ($M^\lambda \subseteq M$ in V).
  - This is not formalizable.
  - Is κ supercompact in V?
- (generically supercompact) There is $j : V \to M \subseteq V[G]$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $j \restriction \lambda \in M$.
  - At least measurable in consistency strength.
- (virtually supercompact) There is $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M \subseteq V$, and $M^\lambda \subseteq M$ in V.
  - Equivalently $M \in V$.
  - Downward absolute to L.
- (generically setwise supercompact) There is $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$ in $V[G]$.
  - Recently defined by Schlicht and Nielsen.
  - (Usuba) Equiconsistent with a virtually extendible cardinal.

Large cardinal embeddings in a forcing extension (continued)

Versions of extendibility

For every $\lambda > \kappa$, in a forcing extension $V[G]$:

- (virtually extendible) There is $j : V_\lambda \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.
  - Downward absolute to L.
- (generically extendible) There is $j : V_\lambda \to V_\beta^{V[G]}$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.
  - Recently defined by Ikegami and Väänänen.
  - (Ikegami, Väänänen) Strong compactness cardinal for second-order Boolean-valued logic.
  - (Usuba) Equiconsistent with a virtually extendible cardinal.

Virtual vs generic large cardinals

Virtual

- set embeddings
- the target M is in V
- the target M has closure in V
- completely ineffable and more
- Downward absolute to L

Generic

- class or set embeddings
- the target M may not be a subset of V
- the target M has closure in $V[G]$
- could be a small cardinal like ω_1
- usually high consistency strength

Virtual embeddings

There is a virtual elementary embedding between first-order structures M and N if they elementarily embed in a forcing extension.

Proposition: There is a virtual isomorphism between the reals $\mathbb{R}$ and the rationals $\mathbb{Q}$.

Proof:

- Force with $\mathrm{Coll}(\omega, \mathbb{R})$ to make $\mathbb{R}$ countable in the forcing extension $V[G]$.
- In $V[G]$, $\mathbb{R}^V$ is a countable dense linear order without endpoints. $\square$

Absoluteness lemma for countable embeddings

Lemma: (Silver) Suppose M and N are first-order structures such that

- M is countable,
- there is an elementary $j : M \to N$.

Suppose W is a transitive (set or class) model of (a large enough fragment of) ZFC such that

- $M, N \in W$,
- M is countable in W.

Then for any finite $\bar{a} \subseteq M$, W has an elementary $j^* : M \to N$ agreeing with j on $\bar{a}$, and (where applicable) $\mathrm{crit}(j) = \mathrm{crit}(j^*)$.

Proof:

- Enumerate $M = \{a_n \mid n < \omega\}$ in W. Let $M \restriction n = \{a_i \mid i < n\}$.
- Let T be the tree of all partial finite isomorphisms

$$f : M \restriction n \to N,$$

  satisfying the requirements, ordered by extension.
- M elementarily embeds into N if and only if T has a cofinal branch.
- T is ill-founded in V, and hence in W. $\square$

Virtual embeddings and collapse extensions

Lemma: Suppose M and N are first-order structures and some set-forcing extension has an elementary $j : M \to N$. Then for every finite $\bar{a} \subseteq M$, $V^{\mathrm{Coll}(\omega, M)}$ has an elementary $j^* : M \to N$ agreeing with j on $\bar{a}$ and (where applicable) $\mathrm{crit}(j) = \mathrm{crit}(j^*)$.

Proof: Suppose a set-forcing extension $V[G]$ has an elementary $j : M \to N$.

- Let $|M|^V = \delta$.
- Consider a further extension $V[G][H]$ by $\mathrm{Coll}(\omega, \delta)$.
- $j \in V[G][H]$ and M is countable in $V[G][H]$.
- $V[H] \subseteq V[G][H]$ has the elementary $j^* : M \to N$ (by Absoluteness lemma). $\square$

Virtual rank-into-rank embeddings in L

Proposition: Assuming $0^{\#}$, L has virtual rank-into-rank embeddings.

Proof:

- Let $\{i_\xi \mid \xi \in \mathrm{Ord}\}$ be the Silver indiscernibles.
- Let $j : L \to L$ be such that $j(i_n) = i_{n+1}$ for $n \in \omega$ and $j(i_\xi) = i_\xi$ for $\xi \geq \omega$.
- Let $i_\gamma = \alpha \gg i_\omega$ so that $j(\alpha) = \alpha$.
- The restriction $j : L_\alpha \to L_\alpha$ is elementary.
- Let $H \subseteq \mathrm{Coll}(\omega, L_\alpha)$ be V-generic.
- $j : L_\alpha \to L_\alpha$ is in the forcing extension $V[H]$.
- In $L[H]$, there is $j^* : L_\alpha \to L_\alpha$ with $\mathrm{crit}(j^*) \leq i_0$ and $j^*(i_\omega) = i_\omega$. $\square$

Observations:

- The supremum of the critical sequence of j^* is at most i_ω.
- Kunen's Inconsistency fails for virtual embeddings!
- Stronger choiceless large cardinals also have consistent virtual versions.

A game characterization of virtual embeddings

Suppose M and N are first-order structures in the same language.
Let $G(M,N)$ be a game played for ω-many steps:

- Player I plays elements $a_n \in M$.
- Player II plays elements $b_n \in N$.
- Players I and II alternate moves.

I	a_0		a_1		a_2		$\dots$		a_n		
II		b_0		b_1		b_2		$\dots$		b_n	

- Player II wins if for every $n \in \omega$ and formula $\varphi(x_0,\dots,x_n)$

$$M \models \varphi(a_0,\dots,a_n) \leftrightarrow N \models \varphi(b_0,\dots,b_n),$$

  the map sending a_i to b_i for $i \leq n$ is a finite partial isomorphism between M and N.
- Otherwise, Player I wins.

A game characterization of virtual embeddings (continued)

Theorem: (Schindler) The following are equivalent.

(1) There is a virtual elementary embedding between M and N.

(2) Player II has a winning strategy in $G(M, N)$.

(3) M elementarily embeds into N in $V^{\mathrm{Coll}(\omega,M)}$.

Proof:

(2) $\Rightarrow$ (3):

- A winning strategy for Player II, remains winning in $V^{\mathrm{Coll}(\omega,M)}$ because no new finite sequences are added.
- In $V^{\mathrm{Coll}(\omega,M)}$, M can be enumerated in an ω-sequence.

(3) $\Rightarrow$ (2): Fix a condition $p \Vdash$ "$\tau : \check{M} \to \check{N}$ is an elementary embedding".

- To every finite $\bar{a}$ from M, associate $p_{\bar{a}} \Vdash \tau(\bar{a}) = \bar{b}$ below p so that: if $\bar{a}'$ extends $\bar{a}$, then $p_{\bar{a}'} \leq p_{\bar{a}}$.
- A winning strategy for Player II: play $\bar{b}$ in response to $\bar{a}$. $\square$

Virtual large cardinals and L

Fill the blank with your favorite virtual large cardinal.

Theorem: (G., Schindler)

- If $0^{\#}$ exists, then every Silver indiscernible is virtually _____.
- Every virtually _____ is downward absolute to L.

Virtually Berkeley and rank-into-rank cardinals

A cardinal δ is Berkeley if for every transitive set M, with $\delta \subseteq M$, and $\gamma < \delta$, there is $j : M \to M$ with $\gamma < \text{crit}(j) < \delta$.

- Inconsistent with ZFC.
- Consistent with ZF?

A cardinal δ is virtually Berkeley if for every transitive set M, with $\delta \subseteq M$, and $\gamma < \delta$, there is a virtual $j : M \to M$ with $\gamma < \text{crit}(j) < \delta$.

Theorem: (Wilson) The least ω-Erdős cardinal is the least virtually Berkeley cardinal.

A cardinal κ is virtually rank-into-rank if there is a virtual $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$.

Theorem: (G., Schindler) The least ω-Erdős cardinal is a limit of virtually rank-into-rank cardinals.

Proof hint: Use indiscernibles.

$C^{(n)}$-extendible cardinals

A cardinal κ is extendible if for every $\kappa < \lambda$, there is $j : V_\lambda \to V_\beta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

Let $C^{(n)} = \{\alpha \in \mathrm{Ord} \mid V_\alpha \prec_{\Sigma_n} V\}$.

A cardinal κ is $C^{(n)}$-extendible if for every $\kappa < \lambda \in C^{(n)}$, there is $j : V_\lambda \to V_\beta$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $\beta \in C^{(n)}$.

Theorem: (Folklore, see Kanamori) We can omit the condition $j(\kappa) > \lambda$.

Proof hint: If there is $j : V_\lambda \to V_\beta$ with $\mathrm{crit}(j) = \kappa$, then either:

- there is $j^* : V_\lambda \to V_{\beta^*}$ with $\mathrm{crit}(j) = \kappa$ and $j^*(\kappa) > \lambda$, or
- there is $j^* : V_{\gamma+2} \to V_{\gamma+2}$ for some γ (Kunen's Inconsistency).

Theorem: (G., Hamkins) A cardinal κ is $C^{(n)}$-extendible if and only if for every Σ_n-definable class A and $\lambda > \kappa$, there is

$$j : (V_\lambda, A \cap \lambda) \to (V_\beta, A \cap \beta)$$

with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

Virtually $C^{(n)}$-extendible cardinals

A cardinal κ is virtually $C^{(n)}$-extendible if for every $\kappa < \lambda \in C^{(n)}$, there is a virtual $j : V_\lambda \to V_\beta$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $\beta \in C^{(n)}$.

A cardinal κ is weakly virtually $C^{(n)}$-extendible if for every $\kappa < \lambda \in C^{(n)}$, there is a virtual $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $\beta \in C^{(n)}$.

Theorem: (G., Schindler) If κ is virtually rank-into-rank, then V_κ is a model of proper class many virtually $C^{(n)}$-extendible cardinals.

Theorem: (G.) If there is a weakly virtually extendible cardinal which is not virtually extendible, then there is a virtually rank-into-rank cardinal.

Corollary: A weakly virtually $C^{(n)}$-extendible cardinal is equiconsistent with a virtually $C^{(n)}$-extendible cardinal.

Vopěnka's Principle

Vopěnka's Principle: Every proper class of first-order structures in the same language has at least two structures which elementarily embed.

Theorem: (Bagaria) The following are equivalent.

- Vopěnka's Principle.
- For every $n < \omega$, there is a proper class of $C^{(n)}$-extendible cardinals.
- For every $n < \omega$, there is a $C^{(n)}$-extendible cardinal.

Virtual Vopěnka's Principle

Virtual Vopenka's Principle: Every proper class of first-order structures in the same language has at least two structures which virtually elementarily embed.

Theorem: (G., Hamkins) Virtual Vopenka's Principle holds if and only if for every $n < \omega$, there is a proper class of weakly virtually $C^{(n)}$-extendible cardinals.

Theorem: (G., Hamkins) It is consistent that Virtual Vopěnka's Principle holds, but there are no virtually supercompact cardinals.

Virtually supercompact cardinals

A cardinal κ is virtually supercompact if for every $\lambda > \kappa$, there is a virtual $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$.

Theorem: (G., Schindler) A cardinal κ is virtually supercompact if and only if it is remarkable.

Theorem: (G., Schindler) A virtually extendible cardinal is a limit of virtually supercompact cardinals.

Virtually strong cardinals

A cardinal κ is virtually strong if for every $\lambda > \kappa$, there is a virtual $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$.

A cardinal κ is weakly virtually strong if for every $\lambda > \kappa$, there is a virtual $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$ and $V_\lambda \subseteq M$.

Theorem: (Nielsen) If there is a weakly virtually strong cardinal which is not virtually strong, then there is a virtually rank-into-rank cardinal.

Corollary: A weakly virtually strong cardinal is equiconsistent with a virtually strong cardinal.

Theorem: (G., Schindler) A cardinal κ is virtually supercompact if and only if it is virtually strong.

(Virtually) Woodin cardinals

A cardinal κ is (virtually) (λ, A)-strong if there is a (virtual)

$$j : (V_\lambda, A \cap V_\lambda) \to (M, \bar{A})$$

with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $\bar{A} \cap V_\lambda = A \cap V_\lambda$.

A cardinal δ is (virtually) Woodin if for every set A, there is $\kappa < \delta$ which is (virtually) $({<}\delta, A)$-strong.

Aside:

A cardinal κ is (virtually) (λ, A)-supercompact if there is a (virtual) $j : (V_\lambda, A) \to (M, \bar{A})$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $\bar{A} \cap V_\lambda = A \cap V_\lambda$.

Theorem: (Perlmutter) Vopěnka's Principle holds if and only if for every class A, there is a $({<}\mathrm{Ord}, A)$-supercompact cardinal (Ord is Woodin for supercompactness).

Theorem: (Dimopolous, G., Nielsen) Virtual Vopěnka's Principle holds if and only if Ord is weakly virtually Woodin.

Virtually measurable cardinals

A cardinal κ is virtually measurable if for every $\lambda > \kappa$, there is a virtual $j : V_\lambda \to M$ with $\mathrm{crit}(j) = \kappa$.

Theorem: (Nielsen) Virtually measurable cardinals are equiconsistent with virtually supercompact cardinals.

Proof: A virtually measurable cardinal κ is weakly virtually strong in L. $\square$

Theorem: (G.) It is consistent that there is a virtually measurable cardinal which is not weakly virtually strong.

Some generic large cardinals

Theorem: (Usuba) The following are equiconsistent.

- virtually extendible cardinal
- (ω_1 or ω_2 is a) generically setwise supercompact cardinal
  - $\kappa > \omega_2$ is generically setwise supercompact implies $0^{\#}$.
- generically extendible cardinal

Some generic large cardinals (continued)

A cardinal κ is faintly strong if for every $\lambda > \kappa$, in a forcing extension $V[G]$, there is $j : V_\lambda \to N$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq N$, and $N \in V[G]$.

A cardinal δ is faintly Woodin if for every set A, there is $\kappa < \delta$ which is faintly $({<}\delta, A)$-strong.

Proposition: A faintly strong cardinal is virtually strong in L. So the two notions are equiconsistent.

Proof: Fix $\bar{\lambda} > \lambda$ and $j : V_{\bar{\lambda}} \to N$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \bar{\lambda}$.
The restriction $j : L_\lambda \to j(L_\lambda) = L_{j(\lambda)}$.

- $L_\lambda \subseteq L_{j(\lambda)}$
- $j(\kappa) > \lambda$
- By Absoluteness lemma, there is $j^* : L_\lambda \to L_{j(\lambda)}$ in L. $\square$

Theorem: (G.) It is consistent that there is a cardinal which is weakly compact (and more), faintly strong, but not virtually strong.

Theorem: (Dimopolous, G., Nielsen) A cardinal κ is virtually Woodin if and only if it is faintly Woodin.

Applications

The model $L(\mathbb{R})$

- Start the L-construction with $\mathbb{R}$ instead of $\emptyset$.
- Satisfies ZF.
- Assuming large cardinals, satisfies the Axiom of Determinacy.

Even though forcing easily changes the theory of V, it is consistent (from large cardinals) that the theory of $L(\mathbb{R})$ cannot be changed by forcing.

Theorem: (Woodin) If there is a supercompact cardinal, then there is a model in which theory of $L(\mathbb{R})$ cannot be changed by forcing.

Theorem: (Schindler) The assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing is equiconsistent with a remarkable (virtually supercompact) cardinal.

Application (continued)

A set of reals is universally Baire if its preimages under all continuous functions from all topological spaces have the Baire property.

- include Σ^1_1-sets and Π^1_1-sets
- Lebesgue measurable
- Baire property
- assuming large cardinals, perfect set property

Theorem: (Schindler, Wilson) The assertion that every universally Baire set has the perfect set property is equiconsistent with a virtually Shelah cardinal.

A wrong version of virtual strongness?

A cardinal κ is (W)-virtually strong if for every $\lambda > \kappa$, in a forcing extension $V[G]$, there is $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda = V_\lambda^M$, but M may be ill-founded above λ.

- defined by Wilson
- κ is (W)-$\kappa + 1$-virtually strong if and only if κ is completely ineffable
- (G.) "much weaker" than virtually supercompact cardinals

Weak Vopěnka's Principle: Technical weakening of Vopěnka's Principle.

Theorem: (Wilson) Weak Vopěnka's Principle holds if and only if for every class A, there is a $({<}\mathrm{Ord}, A)$-strong cardinal (Ord is Woodin).

Theorem: (Wilson) Virtual Weak Vopenka's Principle holds if and only if for every class A, there is a weakly (W)-virtually $({<}\mathrm{Ord}, A)$-strong cardinal (Ord is weakly (W)-virtually Woodin).

Virtual large cardinal chart