Upward Löwenheim-Skolem-Tarski numbers for abstract logics

Victoria Gitman

vgitman@gmail.com
http://victoriagitman.github.io

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First-order logic

Modern mathematics is formalized by means of first-order logic. But what is a logic and what is so special about first-order logic?

Logic

- Assigns a collection of formulas to every language.
- Assigns truth values to formulas for every model.

First-order logic $\mathbb{L}_{\omega,\omega}$

- Formulas: close atomic formulas under conjunctions, disjunctions, negations, quantifiers.
- Truth: Tarski's recursive definition.
- Properties:
  - A language has set-many formulas.
  - A formula can mention finitely much of a language.
  - Compactness: every finitely satisfiable theory has a model.

But first-order logic does not exist outside of mathematics.

A (fragment of a) set-theoretic background is necessary to interpret first-order logic.

- natural numbers
- recursion

Stronger logics require access to more of the set-theoretic background.

Infinitary logics

Add transfinite conjunctions, disjunctions, and quantifier blocks of formulas.

Suppose $\gamma \leq \delta$ are regular cardinals.

Infinitary logics $\mathbb{L}_{\delta,\gamma}$

Close formulas under conjunctions and disjunctions of length $<\delta$ and quantifier blocks of length $<\gamma$.

- A language has set-many formulas.
- A formula can mention $<\delta$-much of a language.

Examples

- $\mathbb{L}_{\omega_1,\omega}$
  - There is a sentence expressing that the natural numbers are standard:
  $$\forall n \in \omega\, [n = 0 \vee n = 1 \vee n = 2 \vee \cdots]$$
  - Compactness fails.
- $\mathbb{L}_{\delta,\omega}$
  - For every ordinal $\xi < \delta$ and formula $\psi(y, x)$, there is a formula $\varphi^{\xi}_{\psi}(x)$ expressing that $(\{y \mid \psi(y, x)\}, \psi) \cong (\xi, \in)$.

Infinitary logics (continued)

Examples (continued)

- $\mathbb{L}_{\omega_1,\omega_1}$
  - For every formula $\psi(x,y)$ there is a sentence $\varphi_\psi^{\mathrm{WF}}$ expressing that the relation given by ψ is well-founded:

$$\neg\exists\, x_0, x_1, \ldots, x_n, \ldots\ [\psi(x_1,x_0) \wedge \psi(x_2,x_1) \wedge \cdots \wedge \psi(x_{n+1},x_n) \wedge \cdots]$$

  - For every formula $\psi(x)$ there is a sentence $\varphi_\psi^{\mathrm{Inf}}$ expressing that $\{x \mid \psi(x)\}$ is infinite:

$$\exists x_0, x_1, \ldots, x_n \ldots \bigwedge_{n,m<\omega} x_n \neq x_m$$

- $\mathbb{L}_{\omega_2,\omega_2}$
  - For every formula $\psi(x)$ there is a sentence φ_ψ expressing that $\{x \mid \psi(x)\}$ is uncountable:

$$\exists x_0, x_1, \ldots, x_\xi \ldots \bigwedge_{\xi,\eta<\omega_1} x_\xi \neq x_\eta$$

Second-order logic $\mathbb{L}^2$

Add second-order quantifiers ranging over all relations on the model.

Expressive power

- The relation given by a formula $\psi(y, x)$ is well-founded: every subset has a least element.
- $\{x \mid \psi(x)\}$ is infinite: there is a bijection with a proper subset.
- $|\{x \mid \psi(x)\}| = |\{y \mid \varphi(y)\}|$
- (Magidor) $(\{y \mid \psi(x, y)\}, \psi) \cong (V_\alpha, \in)$ for some α.
- A group F is free:
  - Suppose F has cardinality δ.
  - F is free if and only if there is a transitive model $M \models \mathrm{ZFC}^-$ of size δ with $F \in M$ which satisfies that F is free.
  - There is a relation E on F such that (F, E)
    - satisfies ZFC^-,
    - is well-founded,
    - has an element isomorphic to F,
    - satisfies that F is free.

Equicardinality logic $\mathbb{L}(I)$

Add a quantifier I such that for all formulas $\psi(x)$ and $\varphi(y)$:

$$Ixy\,\psi(x)\varphi(y) \text{ whenever } |\{x \mid \psi(x)\}| = |\{y \mid \varphi(y)\}|$$

Expressive power

- The natural numbers are standard:
$$\forall n \in \omega\, |\{m \mid m \in n\}| \neq |\{m \mid m \in n+1\}|$$
- $|\{x \mid \psi(x)\}|$ is infinite:
$$\exists y\, [\psi(y) \wedge |\{x \mid \psi(x)\}| = |\{x \mid \psi(x) \wedge x \neq y\}|]$$
- A model is cardinal correct: if κ is a cardinal, then for all $\alpha < \kappa$
$$|\xi \mid \xi < \alpha| \neq |\xi \mid \xi < \kappa|.$$

Relationships

- $\mathbb{L}(I) \subseteq \mathbb{L}^2$

Well-foundeness logic $\mathbb{L}(Q^{\mathrm{WF}})$

Add a new quantifier Q^{WF} such that for all formulas $\psi(x, y)$:

$$Q^{\mathrm{WF}} x, y\, \psi(x, y) \text{ whenever the relation given by } \psi(x, y) \text{ is well-founded.}$$

Relationships

- $\mathbb{L}(Q^{\mathrm{WF}}) \subseteq \mathbb{L}_{\omega_1,\omega_1}$
- $\mathbb{L}(Q^{\mathrm{WF}}) \subseteq \mathbb{L}^2$

Sort logics $\mathbb{L}^{s,n}$

(Väänänen) $\mathbb{L}^{s,n}$

- ω-many sorts
- second-order quantifiers for every sort (extends $\mathbb{L}^2$)
- Sort quantifiers $\tilde{\forall}$ and $\tilde{\exists}$
  - search the set-theoretic universe for a new sort such that there is a relation on the combination of the new and old sorts satisfying a given formula.
  - at most *n*-alternations of sort quantifiers are allowed

Expressive power

- For every formuala $\psi(y,x)$ there is a sentence $\varphi^n_\psi(x)$ expressing that $(\{y \mid \psi(y,x)\}, \psi) \cong (V_\alpha, \in)$ and $V_\alpha \prec_{\Sigma_n} V$ for some α.

Languages

A language τ is a quadruple $(\mathfrak{F}, \mathfrak{R}, \mathfrak{C}, a)$ where:

- $\mathfrak{F}$ are the functions,
- $\mathfrak{R}$ are the relations,
- $\mathfrak{C}$ are the constants,
- $a : \mathfrak{F} \cup \mathfrak{R} \to \omega$ is the arity function.

A τ-structure is a set with interpretations for the functions, relations, and constants in τ.

A renaming f between languages $\tau = (\mathfrak{F}, \mathfrak{R}, \mathfrak{C}, a)$ and $\sigma = (\mathfrak{F}', \mathfrak{R}', \mathfrak{C}', a')$ is an arity-preserving bijection between the functions, relations, and constants.

Given a renaming f, let f^* be the associated bijection between τ-structures and σ-structures.

Logics

A logic is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ of classes satisfying the following conditions.

- $\mathcal{L}$ is a class function which takes a language τ to $\mathcal{L}(\tau)$: the set of all sentences in τ.
- $\models_{\mathcal{L}}$ is the class of all pairs (M, φ) such that M is a τ-structure, $\varphi \in \mathcal{L}(\tau)$ and M satisfies φ according to $\mathcal{L}$.
- If $\tau \subseteq \sigma$ are languages, then $\mathcal{L}(\tau) \subseteq \mathcal{L}(\sigma)$.
- If $\varphi \in \mathcal{L}(\tau)$, $\sigma \supseteq \tau$ are languages, and M is a σ-structure, then $M \models_{\mathcal{L}} \varphi$ if and only if the reduct $M \upharpoonright \tau \models_{\mathcal{L}} \varphi$.
- If $M \cong N$ are τ-structures, then for all $\varphi \in \mathcal{L}(\tau)$ $M \models_{\mathcal{L}} \varphi$ if and only if $N \models_{\mathcal{L}} \varphi$.
- Every renaming f between languages τ and σ induces a bijection $f_* : \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ such that for any τ-structure M and $\varphi \in \mathcal{L}(\tau)$
$$M \models_{\mathcal{L}} \varphi \text{ if and only if } f^*(M) \models_{\mathcal{L}} f_*(\varphi).$$
- There is a least cardinal κ, called the occurrence number of $\mathcal{L}$, such that for every sentence $\varphi \in \mathcal{L}(\tau)$, there is a sub-language τ^* of size less than κ such that $\varphi \in \mathcal{L}(\tau^*)$.

Note: Formulas are accommodated by introducing and interpreting constants.

Strong compactness cardinals

A cardinal κ is a strong compactness cardinal for a logic $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory has a model.

Compactness Theorem: ω is a strong compactness cardinal for first-order logic.

Compactness for $\mathbb{L}_{\kappa,\kappa}$ and $\mathbb{L}_{\kappa,\omega}$

(Tarski) A cardinal κ is strongly compact if every κ-complete filter can be extended to a κ-complete ultrafilter.

- Strongly compact cardinals are stronger than measurable cardinals.
- (Magidor) It is consistent that the least strongly compact cardinal is the least measurable cardinal.

Theorem: (Tarski) The following are equivalent:

- κ is a strong compactness cardinal for $\mathbb{L}_{\kappa,\omega}$.
- κ is a strong compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$.
- κ is strongly compact.

Compactness for $\mathbb{L}_{\omega_1,\omega_1}$ and $\mathbb{L}(Q^{\mathrm{WF}})$

(Magidor) A cardinal κ is ω_1-strongly compact if every κ-complete filter can be extended to a countably complete ultrafilter.

- ω_1-strongly compact cardinals are stronger than measurable cardinals.
- (Magidor) It is consistent that the least ω_1-strongly compact cardinal is the least measurable cardinal.
- (Bagaria, Magidor) It is consistent that the least ω_1-strongly compact cardinal is above the least measurable cardinal.

Theorem: (Magidor) The following are equivalent:

- κ is a strong compactness cardinal for $\mathbb{L}_{\omega_1,\omega_1}$.
- κ is a strong compactness cardinal for $\mathbb{L}(Q^{\mathrm{WF}})$.
- κ is ω_1-strongly compact.

Strong compactness cardinals for $\mathbb{L}^2$ and $\mathbb{L}(I)$

A cardinal κ is extendible if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$, and $j(\kappa) > \alpha$.

Extendible cardinals are stronger than strongly compact cardinals.

Theorem: (Magidor) The least extendible cardinal is the least strong compactness cardinal for $\mathbb{L}^2$.

A cardinal κ is supercompact if for every $\alpha > \kappa$, there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $M^\alpha \subseteq M$.

Theorem: (Boney, Osinski) It is consistent that the least strong compactness cardinal for $\mathbb{L}(I)$ is greater than or equal to the least supercompact cardinal.

Strong compactness cardinals for the sort logics $\mathbb{L}^{s,n}$

$C^{(n)} = \{\alpha \in \mathrm{Ord} \mid V_\alpha \prec_{\Sigma_n} V\}$

(Bagaria) A cardinal κ is $C^{(n)}$-extendible if for every $\alpha > \kappa$ in $C^{(n)}$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$, $\beta \in C^{(n)}$, and $j(\kappa) > \alpha$.

- Extendible cardinals are $C^{(1)}$-extendible.
- $C^{(n)}$-extendible cardinals form a hierarchy.

Theorem: (Boney) The least $C^{(n)}$-extendible cardinal is the least strong compactness cardinal for $\mathbb{L}^{s,n}$.

Universal strong compactness

Vopěnka's Principle holds if for every proper class of first-order structures in the same languages there are two structures which elementarily embed.

Theorem: (Bagaria) Vopěnka's Principle holds if and only if for every $n < \omega$ there is a $C^{(n)}$-extendible cardinal.

Theorem: (Makowsky) Every logic has a strong compactness cardinal if and only if Vopěnka's Principle holds.

Upwards Löwenheim-Skolem-Tarski numbers

Fix a logic $\mathcal{L}$.

The Hanf number of $\mathcal{L}$ is the least cardinal δ such that such that for every language τ and $\mathcal{L}(\tau)$-sentence φ, if a τ-structure $M \models_{\mathcal{L}} \varphi$ has size $\gamma \geq \delta$, then for every cardinal $\overline{\gamma} > \gamma$, there is a τ-structure $\overline{M}$ of size at least $\overline{\gamma}$ such that $\overline{M} \models_{\mathcal{L}} \varphi$.

Theorem: (Folklore) Every logic has a Hanf number.

(Galeotti, Khomskii, Väänänen) The upward Löwenheim-Skolem-Tarski number $\mathrm{ULST}(\mathcal{L})$, if it exists, is the least cardinal δ such that for every language τ and $\mathcal{L}(\tau)$-sentence φ, if a τ-structure $M \models_{\mathcal{L}} \varphi$ has size $\gamma \geq \delta$, then for every cardinal $\overline{\gamma} > \gamma$, there is a τ-structure $\overline{M}$ of size at least $\overline{\gamma}$ such that $\overline{M} \models_{\mathcal{L}} \varphi$ and $M \subseteq \overline{M}$ is a substructure of $\overline{M}$.

The strong upward Löwenheim-Skolem-Tarski number $\mathrm{SULST}(\mathcal{L})$, if it exists, is the least cardinal δ such that for every language τ and every τ-structure M of size $\gamma \geq \delta$, for every cardinal $\overline{\gamma} > \gamma$, there is a τ-structure $\overline{M}$ of size at least $\overline{\gamma}$ such that $M \prec_{\mathcal{L}} \overline{M}$ is an $\mathcal{L}$-elementary substructure of $\overline{M}$.

Upward Löwenheim-Skolem Theorem: ω is the strong upward Löwenheim-Skolem-Tarski number of first-order logic.

Theorem: (Galeotti, Khomskii and Väänänen) If $\mathrm{ULST}(\mathbb{L}^2)$ exists, then for every $n \in \omega$ there is an n-extendible cardinal $\lambda \leq \mathrm{ULST}(\mathbb{L}^2)$.

Compactness and upward Löwenheim-Skolem-Tarski numbers

Proposition: If a logic $\mathcal{L}$ has a strong compactness cardinal κ, then $\mathrm{SULST}(\mathcal{L}) \leq \kappa$.

Proof:

- Fix a τ-structure M of size $\gamma \geq \kappa$.
- Fix a cardinal $\overline{\gamma} > \gamma$.
- Let τ' be the language τ extended by adding $\overline{\gamma}$-many constants $\{c_\xi \mid \xi < \overline{\gamma}\}$.
- Let T be the $\mathcal{L}(\tau')$-theory:
  - $\mathcal{L}$-elementary diagram of M
  - $\{c_\xi \neq c_\eta \mid \xi < \eta < \overline{\gamma}\}$
- T is $<\kappa$-satisfiable (holds in M).
- T has a model. $\square$

Corollary: If Vopěnka's Principle holds, then every logic has a strong upward Löwenheim-Skolem-Tarski number.

The power of substructures in set theory

Suppose $(M, \in)$ and $(N, \in)$ are transitive structures closed under the pairing function.

- $\mathrm{Tr}(\varphi, (x_1, \ldots, x_n))$ if and only if $(M, \in) \models \varphi(x_1, \ldots, x_n)$ is the truth predicate.
- $S(x, y)$ if and only if $y = x \cup \{x\}$ is the successor function.
- $P(x, y, z)$ if and only if $z = (x, y)$ is the pairing function.

Proposition: If $j : (M, \in, S, P, \mathrm{Tr}) \to (N, \in, S, P, \mathrm{Tr})$ is an embedding, then j is elementary.

Upward Löwenheim-Skolem-Tarski numbers for $\mathbb{L}(Q^{\mathrm{WF}})$

Theorem: If κ is a measurable cardinal, then $\mathrm{SULST}(\mathbb{L}(Q^{\mathrm{WF}})) \leq \kappa$.

Proof:

- Fix a τ-structure N of size $\gamma \geq \kappa$.
- Fix a cardinal $\overline{\gamma} > \gamma$.
- Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \overline{\gamma}^{+}$ (sufficiently iterated ultrapower).
- $j(N) \in M$ is a $j(\tau)$-structure, and hence $j\,"\,\tau$-structure.
- $j(N)$ is a τ-structure modulo the renaming which takes τ to $j\,"\,\tau$.
- Let the renaming take φ to $\overline{\varphi}$.
- $\overline{N} = j\,"\,N \subseteq j(N)$ is a τ-substructure of $j(N)$.
- $N \overset{j}{\cong} \overline{N}$
- $\overline{N} \prec_{\mathbb{L}(Q^{\mathrm{WF}})} j(N)$
  - Suppose $\overline{N} \models_{\mathbb{L}(Q^{\mathrm{WF}})} \varphi(j(a))$.
  - $N \models_{\mathbb{L}(Q^{\mathrm{WF}})} \varphi(a)$ via the isomorphism j.
  - $M \models$ "$j(N) \models_{\mathbb{L}(Q^{\mathrm{WF}})} \overline{\varphi}(j(a))$" by elementarity of j.
  - $j(N) \models_{\mathbb{L}(Q^{\mathrm{WF}})} \overline{\varphi}(j(a))$ as a $j(\tau)$-structure (M is well-founded)
  - $j(N) \models_{\mathbb{L}(Q^{\mathrm{WF}})} \varphi(j(a))$ modulo the renaming.
- Since $|N| \geq \kappa$, $|j(N)| \geq j(\kappa) > \overline{\gamma}$. $\square$

Upward Löwenheim-Skolem-Tarski numbers for $\mathbb{L}(Q^{\mathrm{WF}})$ (continued)

Theorem: If $\mathrm{ULST}(\mathbb{L}(Q^{\mathrm{WF}}))$ exists, then it is the least measurable cardinal.

Proof:

- Let $\mathrm{ULST}(\mathbb{L}(Q^{WF})) = \delta$.
- Suffices to show there is a measurable cardinal $\leq\delta$.
- Let $\mathcal{M} = (H_{\delta^+}, \in, \delta, S, P, \mathrm{Tr})$.
- $\mathcal{M} \models_{\mathbb{L}(Q^{\mathrm{WF}})} \varphi$:
  - I am well-founded.
  - δ is the largest cardinal.
  - S is the successor function, P is the pairing function, Tr is the truth predicate.
- Let $\mathcal{N} = (N, \mathsf{E}, \overline{\delta}, \overline{S}, \overline{P}, \overline{\mathrm{Tr}}) \models \varphi$ of size $\gg \delta$ with $\mathcal{M} \subseteq \mathcal{N}$.
- Since $\mathcal{N}$ is well-founded, we can assume:
  - $\mathsf{E} = \in$,
  - N is transitive,
  - $j : H_{\delta^+} \to N$ such that $j(\delta) = \overline{\delta} \gg \delta$.
- j is elementary.
- Let $\mathrm{crit}(j) = \kappa \leq \delta$.
- Use j to derive a κ-complete ultrafilter on κ. $\square$

Upward Löwenheim-Skolem-Tarski numbers for $\mathbb{L}(Q^{\mathrm{WF}})$ (continued)

Corollary: The following are equivalent for a cardinal κ.

- κ is the least measurable cardinal.
- $\kappa = \mathrm{ULST}(\mathbb{L}(Q^{\mathrm{WF}}))$.
- $\kappa = \mathrm{SULST}(\mathbb{L}(Q^{\mathrm{WF}}))$.

Corollary: It is consistent that:

- $\mathrm{ULST}(\mathbb{L}(Q^{\mathrm{WF}})) = \mathrm{SULST}(\mathbb{L}(Q^{\mathrm{WF}}))$ is the least strong compactness cardinal for $\mathbb{L}(Q^{\mathrm{WF}})$. ω_1-strong compact can be the least measurable.
- $\mathrm{ULST}(\mathbb{L}(Q^{\mathrm{WF}})) = \mathrm{SULST}(\mathbb{L}(Q^{\mathrm{WF}}))$ is smaller than the least strong compactness cardinal for $\mathbb{L}(Q^{\mathrm{WF}})$. ω_1-strongly compact can be above the measurable.
- $\mathrm{ULST}(\mathbb{L}(Q^{\mathrm{WF}})) = \mathrm{SULST}(\mathbb{L}(Q^{\mathrm{WF}}))$, but $\mathbb{L}(Q^{\mathrm{WF}})$ doesn't have a strong compactness cardinal. $L[U]$ cannot have an ω_1-strongly compact cardinal.

Upward Löwenheim-Skolem-Tarski numbers for $\mathbb{L}^2$ and $\mathbb{L}^{s,n}$

Observations:

- Targets of extendible embeddings are correct about $\mathbb{L}^2$.
- Targets of $C^{(n)}$-extendible embeddings are correct about $\mathbb{L}^{s,n}$.

Theorem: The following are equivalent for a cardinal κ.

- κ is the least extendible cardinal.
- κ is the least strong compactness cardinal for $\mathbb{L}^2$.
- $\kappa = \mathrm{SULST}(\mathbb{L}^2)$.
- $\kappa = \mathrm{ULST}(\mathbb{L}^2)$.

Theorem: The following are equivalent for a cardinal κ and $n < \omega$.

- κ is the least $C^{(n)}$-extendible cardinal.
- κ is the least strong compactness cardinal for $\mathbb{L}^{s,n}$.
- $\kappa = \mathrm{SULST}(\mathbb{L}^{s,n})$.
- $\kappa = \mathrm{ULST}(\mathbb{L}^{s,n})$.

Corollary: Every logic has an upward Löwenheim-Skolem-Tarski number if and only if Vopěnka's Principle holds.

Tall cardinals

(Hamkins) A cardinal κ is tall if for every $\theta > \kappa$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $M^\kappa \subseteq M$, and $j(\kappa) > \theta$.
A cardinal κ is tall with closure $\lambda \leq \kappa$ if $M^\lambda \subseteq M$, and tall with closure $<\lambda$ if $M^{<\lambda} \subseteq M$.

A cardinal κ is tall pushing up δ if for every $\theta > \delta$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $M^\kappa \subseteq M$, and $j(\delta) > \theta$.
A cardinal κ is tall pushing up δ with closure $\lambda \leq \kappa$ if $M^\lambda \subseteq M$, and tall with closure $<\lambda$ if $M^{<\lambda} \subseteq M$.

A cardinal δ is supreme for tallness if for all $\lambda < \delta$, there is a cardinal $\lambda < \kappa \leq \delta$ that is tall pushing up δ with closure λ.

- (Gitik) Tall cardinals are stronger than measurable cardinals. equiconsistent with strong cardinals
- Strongly compact cardinals are stronger than tall cardinals.
- (Hamkins) If κ is tall with closure $<\kappa$, then κ is tall.
- (Hamkins) It is consistent that there is a tall cardinal κ pushing up some $\delta > \kappa$, which is not tall.
- A limit of tall cardinals is supreme for tallness.

Question: Are tall cardinals and tall cardinals pushing up δ equiconsistent?

Upward Löwenheim-Skolem-Tarski numbers for $\mathbb{L}_{\kappa,\kappa}$

Observation: Targets of tall with closure $<\lambda$ embeddings are correct about $\mathbb{L}_{\lambda,\lambda}$.

Proposition: $\mathrm{ULST}(\mathbb{L}_{\kappa,\kappa}) \geq \kappa$.

Theorem: If there is a tall cardinal κ pushing up δ with closure $<\lambda$, then $\mathrm{SULST}(\mathbb{L}_{\lambda,\lambda}) \leq \delta$. In particular, if κ is tall, then $\mathrm{SULST}(\mathbb{L}_{\kappa,\kappa}) = \mathrm{ULST}(\mathbb{L}_{\kappa,\kappa}) = \kappa$.

Theorem: If $\mathrm{SULST}(\mathbb{L}_{\lambda,\lambda}) = \delta$, then there is a tall cardinal $\lambda \leq \kappa \leq \delta$ pushing up δ with closure $<\lambda$. In particular, if $\mathrm{SULST}(\mathbb{L}_{\kappa,\kappa}) = \kappa$, then κ is tall.

Corollary: It is consistent that $\mathrm{ULST}(\mathbb{L}_{\kappa,\kappa}) = \mathrm{SULST}(\mathbb{L}_{\kappa,\kappa}) = \kappa$, but κ is not a strong compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$.

Theorem: If δ is supreme for tallness, then $\mathrm{ULST}(\mathbb{L}_{\lambda,\lambda}) \leq \delta$ exists for every regular $\lambda \leq \delta$. In particular, if δ is regular, then $\mathrm{ULST}(\mathbb{L}_{\delta,\delta}) = \delta$.

Theorem: If $\mathrm{ULST}(\mathbb{L}_{\lambda,\lambda}) = \lambda$, then λ is supreme for tallness.

Upward Löwenheim-Skolem-Tarski numbers for $\mathbb{L}_{\kappa,\kappa}$ (continued)

Theorem: It is consistent that λ is inaccessible, $\mathrm{ULST}(\mathbb{L}_{\lambda,\lambda})$ exists, but $\mathrm{SULST}(\mathbb{L}_{\lambda,\lambda})$ does not exist.

Proof sketch: Use forcing to produce a model with:

- An inaccessible λ that is a limit of tall cardinals.
- No measurable cardinals $\geq \lambda$.

Theorem: It is consistent that λ is inaccessible and $\mathrm{ULST}(\mathbb{L}_{\lambda,\lambda}) < \mathrm{SULST}(\mathbb{L}_{\lambda,\lambda})$.

Proof sketch: Use forcing to produce a model with:

- An inaccessible λ that is a limit of tall cardinals.
- λ is not tall.
- There is a tall cardinal above λ.

$\mathbb{L}(I)$ and well-foundedness

Let ZFC^* be a sufficiently large finite fragment of ZFC:

- the ordinals form a well-ordered class,
- for every ordinal α there is a sequence of cardinals of order-type α,
- the sets are the union of the von Neumann hierarchy.

Theorem: (Goldberg) If $(M, \mathrm{E}) \models \mathrm{ZFC}^*$ is cardinal correct, then it is well-founded.

Proof:

- Suffices to show that every ordinal is well-founded.
- Fix an ordinal α in M.
- Let $\{\kappa_\xi \mid \xi \,\mathrm{E}\, \alpha\}$ be a sequence of cardinals of order-type α in M.
- In V $|\kappa_\xi| < |\kappa_\eta|$ for all $\xi \,\mathrm{E}\, \eta \,\mathrm{E}\, \alpha$ (cardinal correctness).
- α is well-founded. $\square$

Cardinal correct extendible cardinals

A cardinal κ is cardinal correct extendible if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to M$ with $\operatorname{crit}(j) = \kappa$, M cardinal correct, and $j(\kappa) > \alpha$.
A cardinal κ is weakly cardinal correct extendible if we remove $j(\kappa) > \alpha$.

A cardinal κ is cardinal correct extendible pushing up δ if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to M$ with $\operatorname{crit}(j) = \kappa$, M cardinal correct, and $j(\delta) > \alpha$.

Theorem:

- If κ is cardinal correct extendible, then κ is strongly compact.
- If κ is weakly cardinal correct extendible, then either κ is strongly compact or there is an inaccessible α such that V_α satisfies that there is a strongly compact cardinal.

Theorem: It is consistent that the least supercompact cardinal is not not cardinal correct extendible.

Theorem: (Osinski, Poveda) It is consistent that the least supercompact cardinal is cardinal correct extendible.

Questions:

- Are cardinal correct extendible cardinals weaker than extendible cardinals?
- Can the least measurable be the least cardinal correct extendible?
- Are weakly cardinal correct extendible cardinals equivalent to cardinal correct extendible cardinals?

Upward Löwenheim Skolem numbers for $\mathbb{L}(I)$

Observation: The target of a cardinal correct extendible embedding is cardinal correct.

Theorem: If there is a cardinal correct extendible cardinal κ pushing up δ, then $\mathrm{SULST}(\mathbb{L}(I)) \leq \delta$.

Theorem: If $\mathrm{ULST}(\mathbb{L}(I))$ exists, then there are $\kappa \leq \gamma$ such that κ is cardinal correct extendible pushing up γ.

Theorem: It is consistent that $\mathrm{ULST}(\mathbb{L}(I))$ is strictly above the least supercompact cardinal.

Question: If $\mathrm{SULST}(\mathbb{L}(I)) = \delta$, is there a cardinal correct extendible $\kappa \leq \delta$ pushing up δ?